Wave Equation

Mathematical manipulation of Faraday's law and Ampere-Maxwell law leads directly to a wave equation for the electric and magnetic field.

Wave equation for the electric field

we know from Maxwell's curl equations for the electric field

$$\nabla\times\vec{E} = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t} \quad \dots(2.1)$$

$$\nabla\times\vec{H} = \sigma E + \frac{\partial D}{\partial t} \quad \dots(2.2)$$

Where the conduction current density $\boldsymbol{J}$ in a given medium is defined by $\boldsymbol{J} = \boldsymbol{\sigma E}$ and $\boldsymbol{\sigma}$ is the conductivity of the medium in S/m (℧/m). Taking the curl of eqn. (2.1)

$$\nabla\times(\nabla\times\vec{E}) = -\mu\frac{\partial}{\partial t}(\nabla\times H)$$

$$\nabla\times(\nabla\times E) = -\mu\frac{\partial}{\partial t}\left(\sigma E + \frac{\epsilon\partial E}{\partial t}\right)$$

By applying vector identity:

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla.\mathrm{E}) - \nabla^2 E$$

It is assumed that $\sigma = 0$ and $\nabla.\mathrm{E} = 0$ as $\rho_v = 0$ from equation $\nabla.\vec{D} = \rho_v$

as **D = ϵ E**

$$-\nabla^2 E = -\mu\epsilon\frac{\partial^2 E}{\partial t^2}$$

$$\boxed{\nabla^2 E = \mu\epsilon\frac{\partial^2 E}{\partial t^2}} \dots(2.3)$$

Wave equation for the magnetic field

Similarly taking the curl of the following equation

$$\nabla\times\vec{H} = \epsilon\frac{\partial E}{\partial t}$$

$$\nabla\times(\nabla\times\overrightarrow{H}) = \epsilon\frac{\partial}{\partial t}(\nabla\times\overrightarrow{E})$$

$$= -\mu\epsilon\frac{\partial^2 H}{\partial t^2}$$

$$\nabla.(\nabla.\overrightarrow{H}) - \nabla^2 H = -\mu\epsilon\frac{\partial^2 H}{\partial t^2}$$

AS $\nabla.\overrightarrow{H} = 0$ **from** $\nabla.B = 0$

$$-\nabla^2 H = -\mu\epsilon\frac{\partial^2 H}{\partial t^2}$$

$$\boxed{\nabla^2 H = \mu\epsilon\frac{\partial^2 H}{\partial t^2}} \quad \ldots \quad (2.4)$$

It should be noted that the "double del" or "del squared" is a scalar product that is,

$$\nabla \cdot \nabla = \nabla^2$$

which is a second-order operator in three different coordinate systems. In rectangular (cartesian) coordinates,

$$\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In cylindrical (circular) coordinates,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

In spherical coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial_r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Eqns (2. 3) and (2.4) are the Helmholtz wave equation satisfied by electric and magnetic fields.

Solution of wave equation

Both electric and magnetic fields are the functions of x,y,z and time (four variables). All the components of E (E_x, E_y, E_z) and H (H_x, H_y, H_z) will be satisfying the wave equation.

Let us assume that there is no spatial variation of y and x, $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$. Let us assume that E is varying only in the direction of propagation i.e. (y-direction)

$$\nabla^2 E_x - \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} = 0$$

To simplify Maxwell's equations by writing them in terms of phasors just like we use in circuit analysis, then

$$E_x = E_{x\circ} e^{jwt}$$

Then $$\frac{\partial^2 E_x}{\partial t^2} = -w^2 E_x$$

$$\nabla^2 E_x + \mu\epsilon w^2 E_x = 0$$

$$\boxed{\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \mu\epsilon w^2 E_x = 0} \quad \dots \ (2.5)$$

Equ. (2.5) can easily be solved if the electric field is considered to vary only in one direction i. e. direction of propagation. Such a wave is called uniform plane wave characterized by uniform E_x (uniform in magnitude as well as in phase) over an infinite plane surface perpendicular to the direction of propagation.

$$E_x = E_{x\circ} e^{j(wt-\beta z)} \quad \dots \ (2.6)$$

Where:

z → direction of propagation

w → angular frequency

$\boldsymbol{\beta} \rightarrow$ phase constant

Hence Maxwell's Equations (phasor form) will be

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$
$$\nabla \times \mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E}$$

Plane Wave

A plane wave is a wave whose phase is constant over a set of planes

Uniform Plane Wave

A plane wave is a wave whose phase and magnitude are both constant. A spherical wave in free space is a uniform plane wave. Electromagnetic waves in free space are typical uniform Plane Wave.

A. Wave propagation in free space

Assume an electric field E_x **propagates** in the z-direction, then:

$$\frac{\partial^2 \overline{E}}{\partial x^2} + \frac{\partial^2 \overline{E}}{\partial y^2} + \frac{\partial^2 \overline{E}}{\partial z^2} = -\omega^2 \mu_o \varepsilon_o \overline{E}$$

Since only component of the electric field exists in the x-direction; then:

$$\frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu_o \varepsilon_o E_x$$

The general solution of the 2nd order differential equation is:

The general solution of the 2nd order differential equation is:

$$E_x = Ae^{-j\omega z\sqrt{\mu_o \varepsilon_0}}$$

Introducing the time parameter, and use of trigonometric form, the real part will be:

$$E_x = A\cos(\omega t - \omega z\sqrt{\mu_o\varepsilon_o})$$

$$\therefore E_x = A\cos[\omega(t - z\sqrt{\mu_o\varepsilon_o})]$$

at $z = 0$, and $t = 0$, $E_x = E_{xo}$

$$E_x = E_{xo}\cos[\omega(t - z\sqrt{\mu_o\varepsilon_o})]$$

$$\eta = \frac{E_x}{H_y} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi = 377\Omega$$

η is intrinsic impedance of the medium

and

$$c = \frac{1}{\sqrt{\mu_o\varepsilon_o}},$$

$$c = \frac{\omega}{\beta}$$

c or v_p is phase velocity

For free space $\alpha = 0$, $\sigma = 0$ *and* $\beta = w\sqrt{\mu_\circ\varepsilon_\circ}$

α is the attenuation constant (Neper/m) or Np/m

We define γ as the propagation constant

$$\gamma = \alpha + j\beta$$

$$\therefore \gamma = j\,w\sqrt{\mu_\circ\varepsilon_\circ}$$

The distance between corresponding adjacent points on the wave known as the wave length **(λ)**

$$\lambda = \frac{2\pi}{\beta}$$

B. Wave propagation in lossless dielectric

In lossless dielectric $\boldsymbol{\sigma = 0}$, $\boldsymbol{\varepsilon = \varepsilon_\circ \varepsilon_r}$, $\boldsymbol{\mu_\circ \mu_r}$

$$\boldsymbol{\alpha = 0}, \quad \boldsymbol{\beta = w\sqrt{\mu\varepsilon}}$$

$$v_p = \frac{w}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}\,, \qquad \boldsymbol{\eta} = \sqrt{\frac{\mu}{\varepsilon}} = \boldsymbol{377}\sqrt{\frac{\mu_r}{\varepsilon_r}}\,\boldsymbol{\Omega}$$

$$\boldsymbol{\gamma = \alpha + j\beta}$$

$$\therefore \boldsymbol{\gamma = j\,w\sqrt{\mu\varepsilon}}$$

C. Wave propagation in lossy dielectric $(\boldsymbol{\sigma \neq 0})$

In the lossy dielectric medium, we have (σ) and the charge density:

$$\overline{J} = \sigma\overline{E}$$

$$\therefore \nabla \times \overline{H} = \overline{J} + j\omega\varepsilon\overline{E} = \sigma\overline{E} + j\omega\varepsilon\overline{E} = (\sigma + j\omega\varepsilon)\overline{E}$$

$$\gamma = \pm j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

The **intrinsic impedance** is now a complex quantity:

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{1}{\sqrt{1 - j\left(\sigma/\omega\varepsilon\right)}}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

$$v_p = \frac{w}{\beta} \qquad and \qquad \lambda = \frac{2\pi}{\beta}$$

D. Wave propagation in good conductors & skin depth

The general form of the propagation constant γ is:

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

In conductors; $\frac{\sigma}{\omega\varepsilon} >> 1$, this results in:

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{- j\frac{\sigma}{\omega\varepsilon}}$$

$$\therefore \gamma = j\sqrt{- j\omega\mu\sigma}$$

$$\alpha = \beta = \sqrt{\frac{w\mu\sigma}{2}}$$

The exponential factor $e^{-\alpha z}$ of the traveling wave becomes e^{-1} **=0.368** when $z = \frac{1}{\sqrt{\frac{w\mu\sigma}{2}}}$

The inverse of the attenuation constant for good conductors is defined as the *skin depth* δ_s . The skin depth defines the distance over which a plane traveling in a good conductor wave decays by an amount of e^{-1} = 0.368.

$$\delta_s = \frac{1}{\sqrt{\frac{w\mu\sigma}{2}}} = \frac{1}{\alpha}$$

The **intrinsic impedance** of a good conductor is given by

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1 + j)R_s$$

Where R_s is the surface impedance and is given by:

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$